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Isomorphisms of Some Lattices Obtained from Morita Context of Semigroups Using Fuzzy Notions

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Abstract. There are four algebraic structures in a Morita context $< S, T,_S P_T,_T Q_S, \theta, \phi >$, two semigroups S, T and two bioperands P, Q. There is nice interplay among these components which is evident from the study accomplished by Laan et al. [11], Gupta et al. [8] and Sardar et al. [18]. Among other things, connection between different types of lattices of various pairs of components of a Morita context have been obtained in those papers. The main purpose of this paper is to use fuzzy notions in order to explore some more interplays among the different components of a Morita context of semigroups. In this endeavour we have obtained various types of lattices along with isomorphism between various relevant pairs of them.

Keywords: Semigroups with weak local units; Strong Morita equivalence; Fuzzy ideal; Fuzzy congruence; Fuzzy semilattice congruence.

1. Introduction

Morita theory for rings [14] plays an important role in the study of structure theory of rings. Following the tradition of taking impetus from the multiplicative structure of rings in order to develop the theory of semigroups Knauer [9] and Banschewski [2] independently introduced the Morita theory for monoids and obtained similar results. They also obtained connection between Morita equivalences of monoids and Morita context of Monoids. Talwar [21, 22] extended Morita theory to arbitrary semigroups. Subsequently there has been some more works on Morita theory of semigroups [3, 8, 11, 18] as well as of semirings [5, 6, 7, 17, 19]. One aspect of the study of Morita theory of semigroups is to explore nice interplays among various components of Morita context of semigroups. In this connection we may refer to some of the results viz. [8, Theorems 2.12, 4.10, 5.3], [11, Theorems 3, 6], [18, Theorems 2.1, 2.5].

On the other hand after the introduction of fuzzy sets by Zadeh [24] many branches of mathematics have used fuzzy notions in various ways. Algebra is no exception. In this connection we may refer to at least two monographs by Mordeson et al. [12, 13] in order to highlight the sustained research interest in the area of fuzzification of algebra. This trend together with nice interplay among various components of a Morita context of semigroups as mentioned in the previous paragraph has motivated us to use fuzzy notions in order to explore some more interplays among the components S, T, P, Q of a Morita context $< S, T, {}_S P_T, {}_T Q_S, \theta, \phi >$ of semigroups (*cf.* Definition 2.2).

In this endeavour we have used various fuzzy notions viz. fuzzy ideals, fuzzy suboperands, l-prime fuzzy ideals, l-prime fuzzy suboperands, fuzzy congruences, fuzzy operand equivalences, fuzzy semilattice congruences to relevant components of a Morita context of semigroups and obtained four lattices in each case. We have then obtained isomorphism between any two lattices of the four lattices in each case. This has resulted into four main theorems viz. Corollary 3.2 and Theorems 3.6, 3.8 and 3.10.

2. Preliminaries

This section contains some basic notions which will be used in the sequel.

Definition 2.1. [11] *A semigroup S is said to have weak local units if for every $s \in S$ there exist $u_s,\ v_s \in S$ such that $u_s s = s = s v_s$ and have common joint* *weak local units if for all* $s, s' \in S$ *there exist* $u,\ v \in S$ *such that* $us = s = sv$ *and* $us' = s' = s'v$.

Definition 2.2. [21] *A Morita context is a six-tuple where* S, T *are semigroups,* ${}_SP_T$ *and* ${}_TQ_S$ *are respectively a bioperand over* S *and* T *and a bioperand over* T *and* S, *and* $\theta:\ {}_S(P \otimes_T Q)_S \rightarrow\ {}_SS_S$ *and* $\phi:\ {}_T(Q \otimes_S P)_T \rightarrow\ {}_TT_T$ *are bioperand homomorphisms such that* $\theta(p \otimes q)p' = p\phi(q \otimes p')$ *and* $\phi(q \otimes p)q' = q\theta(p \otimes q')$ *for every* $p, p' \in P$ *and* $q, q' \in Q$. *Bioperands and biacts are synonymous but here we call them bioperands to keep uniformity with the terminology fuzzy operand of [15].*

Moreover, a Morita context is called unitary if ${}_SP_T$ *and* ${}_TQ_S$ *are unitary bioperands.*

Definition 2.3. [22] *Semigroups* S *and* T *are said to be strongly Morita equivalent if there exists a unitary Morita context* $< S, T, {}_SP_T, {}_TQ_S, \theta, \phi >$ *with* θ *and* ϕ *surjective.*

For more details on Morita equivalence of semigroups the readers are referred to [11].

Definition 2.4. [24] *A fuzzy subset* μ *of a non-empty set* X *is a function* $\mu : X \rightarrow [0, 1]$.

Definition 2.5. [10] *A fuzzy subset* μ *of a semigroup* S *is called a fuzzy left (right) ideal of* S *if* $\mu(xy) \geq \mu(y)$ *(resp.* $\mu(xy) \geq \mu(x)$*) for all* $x, y \in S$. *A fuzzy subset of* S *is called a fuzzy ideal if it is both fuzzy right ideal and fuzzy left ideal.*

Definition 2.6. [23] *A fuzzy ideal* μ *of a semigroup* S *is called an* l*-prime fuzzy ideal if* $\mu(xy) = \max\{\mu(x), \mu(y)\}$ *for all* $x, y \in S$.

Definition 2.7. [24] *A fuzzy relation* σ *on a non-empty set* X *is a function* $\sigma : X \times X \rightarrow [0, 1]$.

Definition 2.8. [25] *A fuzzy relation* σ *on a non-empty set* X *is said to be a fuzzy equivalence relation on* X *if it satisfies the following for all* $x, y \in X$:

(1) *Fuzzy reflexive:* $\sigma(x, x) = 1$.
(2) *Fuzzy symmetric:* $\sigma(x, y) = \sigma(y, x)$.
(3) *Fuzzy transitive:* $\sigma(x, y) \geq \bigvee_{z \in X}\{\sigma(x, z) \wedge \sigma(z, y)\}$, *i.e.,* $\sigma \circ \sigma \subseteq \sigma$.

Definition 2.9. [16] *A fuzzy equivalence relation* μ *on a semigroup* S *is called a fuzzy congruence if* $\mu(ax, ay) \geq \mu(x, y)$ *and* $\mu(xa, ya) \geq \mu(x, y)$ *for all* $x, y, a \in S$.

Definition 2.10. [1] *Let S be a semigroup and M be a right operand (left operand) over S. Then a function $\mu : M \to [0,1]$ is called fuzzy right suboperand (fuzzy left suboperand) of M if $\mu(ms) \geq \mu(m)$ $(\mu(sm) \geq \mu(m))$ for all $m \in M$, $s \in S$.*

Let S and T be two semigroups and M be a bioperand over S and T. Then a function $\mu : M \to [0,1]$ is called fuzzy bisuboperand of M if it is both fuzzy right suboperand and fuzzy left suboperand of M.

Definition 2.11. [15] *Let S be a semigroup, M a right operand over S. Then a fuzzy equivalence relation σ on M is called a fuzzy operand equivalence on M if $\sigma(xs, ys) \geq \sigma(x, y)$ for all $x, y \in M$, $s \in S$.*

In the rest of the paper, for simplicity, we use the term fuzzy suboperand instead of fuzzy bisuboperand.

3. Main Results

For a Morita context $< S, T, {}_S P_T, {}_T Q_S, \theta, \phi >$ of semigroups, $FID(S)$, $FID(T)$ respectively denote the set of all ideals of S and T; $FSO(P)$, $FSO(Q)$ respectively denote the set of all fuzzy suboperands of ${}_S P_T$ and ${}_T Q_S$.

Theorem 3.1. *Suppose S and T are semigroups with weak local units. If S and T are strongly Morita equivalent via the Morita context $< S, T, {}_S P_T, {}_T Q_S, \theta, \phi >$, then there is an inclusion preserving bijection between any two of the following sets.*

(1) $FID(S)$.
(2) $FID(T)$.
(3) $FSO(P)$.
(4) $FSO(Q)$.

Proof. We first show the inclusion preserving bijection between $FID(S)$ and $FSO(P)$.

For a fuzzy ideal μ of S, we define $\mu^+ : P \to [0,1]$ by

$$\mu^+(p) := \inf_{q \in Q} \mu(\theta(p \otimes q)),$$

and for a fuzzy suboperand f of P, we define $f^{+'} : S \to [0,1]$ by

$$f^{+'}(s) := \inf_{p \in P} f(sp).$$

Let $s \in S,\ p \in P,\ t \in T$. Then

$$\begin{aligned}
\mu^+(sp) &= \inf_{q \in Q} \mu(\theta(sp \otimes q)) = \inf_{q \in Q} \mu(s\theta(p \otimes q)) \\
&\geq \inf_{q \in Q} \mu(\theta(p \otimes q)) \quad \text{(as } \mu \text{ is a fuzzy ideal of } S\text{)} \\
&= \mu^+(p)
\end{aligned}$$

and

$$\begin{aligned}\mu^+(pt) &= \inf_{q\in Q} \mu(\theta(pt\otimes q))\\ &= \inf_{q\in Q} \mu(\theta(p\otimes tq))\\ &\geq \inf_{q\in Q} \mu(\theta(p\otimes q)) = \mu^+(p).\end{aligned}$$

So μ^+ is a fuzzy suboperand of P.

Now let $x, y \in S$. Then

$$\begin{aligned}f^{+'}(xy) &= \inf_{p\in P} f(xyp)\\ &\geq \inf_{p\in P} f(yp) = f^{+'}(y).\end{aligned}$$

Also

$$\begin{aligned}f^{+'}(xy) &= \inf_{p\in P} f(xyp)\\ &\geq \inf_{p\in P} f(xp) \quad = f^{+'}(x).\end{aligned}$$

So $f^{+'}$ is a fuzzy ideal of S.

Now let us define $\Theta^+ : FID(S) \to FSO(P)$ by

$$\Theta^+(\mu) := \mu^+$$

and $\Theta^{+'} : FSO(P) \to FID(S)$ by

$$\Theta^{+'}(f) := f^{+'}.$$

Then for $s \in S$

$$\begin{aligned}(\Theta^{+'}\circ\Theta^+)(\mu)(s) &= (\mu^+)^{+'}(s)\\ &= \inf_{p\in P} \mu^+(sp)\\ &= \inf_{p\in P}\inf_{q\in Q} \mu(\theta(sp\otimes q))\\ &= \inf_{p\in P}\inf_{q\in Q} \mu(s\theta(p\otimes q)) = \mu(s).\end{aligned}$$

The last equality holds since μ is a fuzzy ideal of S and p, q can be chosen such that $\theta(p\otimes q)$ is a weak right local unit of s.

Again for $p \in P$

$$\begin{aligned}(\Theta^+\circ\Theta^{+'})(f)(p) &= (f^{+'})^+(p)\\ &= \inf_{q\in Q} f^{+'}(\theta(p\otimes q))\\ &= \inf_{q\in Q}\inf_{p'\in P} f(\theta(p\otimes q)p')\\ &= \inf_{q\in Q}\inf_{p'\in P} f(p\phi(q\otimes p')) = f(p).\end{aligned}$$

The last equality holds since f is a fuzzy suboperand of P and p', q can be chosen such that $p\phi(q \otimes p') = p$. Hence Θ^{+} and $\Theta^{+'}$ are mutually inverse bijections.

Now let $\mu_1, \mu_2 \in FID(S)$. Then $\mu_1 \subseteq \mu_2$ implies

$$\begin{aligned} \mu_1^{+}(p) &= \inf_{q \in Q} \mu_1(\theta(p \otimes q)) \\ &\leq \inf_{q \in Q} \mu_2(\theta(p \otimes q)) = \mu_2^{+}(p) \end{aligned}$$

for all $p \in P$. Hence Θ^{+} preserves order.

In order to complete the proof we only prescribe below the functions which give rise to the relevant bijections. We omit the details as they are the relevant modifications of the arguments applied above.

For a fuzzy ideal μ of T we define $\Theta^{*} : FID(T) \to FSO(P)$ by

$$\Theta^{*}(\mu) := \mu^{*}$$

and for a fuzzy suboperand f of P we define $\Theta^{*'} : FSO(P) \to FID(T)$ by

$$\Theta^{*'}(f) := f^{*'},$$

where $\mu^{*} : P \to [0,1]$ is defined by

$$\mu^{*}(p) := \inf_{q \in Q} \mu(\phi(q \otimes p)),$$

and $f^{*'} : T \to [0,1]$ is defined by

$$f^{*'}(t) := \inf_{p \in P} f(pt).$$

For a fuzzy ideal μ of S we define ${}^{+}\Theta : FID(S) \to FSO(Q)$ by

$${}^{+}\Theta(\mu) :=^{+} \mu,$$

and for a fuzzy suboperand f of Q we define ${}^{+'}\Theta : FSO(Q) \to FID(S)$ by

$${}^{+'}\Theta(f) :=^{+'} f,$$

where ${}^{+}\mu : Q \to [0,1]$ is defined by

$${}^{+}\mu(q) := \inf_{p \in P} \mu(\theta(p \otimes q)),$$

and ${}^{+'}f : S \to [0,1]$ is defined by

$${}^{+'}f(s) := \inf_{q \in Q} f(qs).$$

For a fuzzy ideal μ of T we define ${}^{*}\Theta : FID(T) \to FSO(Q)$ by

$${}^{*}\Theta(\mu) :=^{*} \mu,$$

and for a fuzzy suboperand f of Q we define ${}^{*'}\Theta : FSO(Q) \to FID(T)$ by

$$ {}^{*'}\Theta(f) := {}^{*'} f, $$

where ${}^{*}\mu : Q \to [0,1]$ is defined by

$$ {}^{*}\mu(q) := \inf_{p \in P} \mu(\phi(q \otimes p)), $$

and ${}^{*'}f : T \to [0,1]$ is defined by ${}^{*'}f(t) := \inf_{q\in Q} f(tq)$. ∎

(With the same notations as above) It is a matter of routine verification that $FID(S)$, $FID(T)$, $FSO(P)$, $FSO(Q)$ form lattices (with respect to union of fuzzy sets as the *join* and intersection of fuzzy sets as the *meet*). Then in view of [4, Prop. 2.19] we obtain from the above theorem the following corollary.

Corollary 3.2. *Suppose S and T are semigroups with weak local units. If S and T are strongly Morita equivalent via the Morita context $< S, T,_S P_T,_T Q_S, \theta, \phi >$, then any two of the following lattices are isomorphic:*

(1) *the lattice $FID(S)$ of fuzzy ideals of S;*
(2) *the lattice $FID(T)$ of fuzzy ideals of T;*
(3) *the lattice $FSO(P)$ of fuzzy suboperands of ${}_SP_T$;*
(4) *the lattice $FSO(Q)$ of fuzzy suboperands of ${}_TQ_S$.*

Now we use the notions of l-prime fuzzy ideals (*cf.* Definition 2.6) and l-prime fuzzy suboperands (Definition 3.3 follows) and obtain four lattices associated respectively with each component of a Morita context of strongly Morita equivalent semigroups.

Definition 3.3. *Let S and T be two semigroups and M be a bioperand over S and T and f be a non-empty fuzzy subset on M. Then f is called an l-prime fuzzy suboperand on M if $f(sm) = (\underset{m'\in M}{\wedge} f(sm')) \vee f(m)$ and $f(mt) = f(m) \vee (\underset{m'\in M}{\wedge} f(m't))$ for all $s \in S, t \in T, m \in M$.*

Theorem 3.4. *Let S and T be strongly Morita equivalent via the Morita context $< S, T,_S P_T,_T Q_S, \theta, \phi >$ and f be a non-empty fuzzy subset on M. Then f is an l-prime fuzzy suboperand of P if and only if $\inf\limits_{q\in Q} f(\theta(p\otimes q)p') = \inf\limits_{q\in Q} f(p\phi(q\otimes p')) = f(p) \vee f(p')$ for all $p, p' \in P$.*

Proof. Let f be an l-prime fuzzy suboperand of P. Then for $p, p' \in P$

$$ \begin{aligned} \inf_{q\in Q} f(\theta(p\otimes q)p') &= \inf_{q\in Q} \Big(\inf_{p''\in P} f(\theta(p\otimes q)p'') \vee f(p') \Big) \\ &= \inf_{q\in Q} \Big(\inf_{p''\in P} f(p\phi(q\otimes p'')\Big) \vee f(p') \\ &= f(p) \vee f(p'). \end{aligned} $$

The last equality holds as q, p'' can be chosen such that $p\phi(q \otimes p'') = p$.

Conversely, suppose $\inf_{q\in Q} f(\theta(p \otimes q)p') = \inf_{q\in Q} f(p\phi(q \otimes p')) = f(p) \vee f(p')$ for all $p, p' \in P$. Then for $s \in S$ and $p \in P$,

$$\begin{aligned}(\inf_{p'\in P} f(sp')) \vee f(p) &= \inf_{p'\in P}\big(f(sp') \vee f(p)\big)\\ &= \inf_{p'\in P}\Big(\inf_{q'\in Q} f(\theta(sp' \otimes q')p)\Big)\\ &= \inf_{\substack{p'\in P\\ q'\in Q}} f(s\theta(p' \otimes q')p)\\ &= f(sp).\end{aligned}$$

The last equality holds as p', q' can be chosen such that $\theta(p' \otimes q')$ is a weak right local unit of s.

Also for $t \in T$ and $p \in P$,

$$\begin{aligned}f(p) \vee (\inf_{p''\in P} f(p''t)) &= \inf_{p''\in P}\big(f(p) \vee f(p''t)\big)\\ &= \inf_{\substack{p''\in P\\ q''\in Q}} f(p\phi(q'' \otimes p''t))\\ &= \inf_{\substack{p''\in P\\ q''\in Q}} f(p\phi(q'' \otimes p'')t)\\ &= f(pt).\end{aligned}$$

The last equality holds as p'', q'' can be chosen such that $\phi(q'' \otimes p'')$ is a weak left local unit of t. Hence f is an l-prime fuzzy suboperand of P. ∎

Remark 3.5. Let S and T be strongly Morita equivalent via the Morita context $< S, T, {}_SP_T, {}_TQ_S, \theta, \phi >$ and f be a non-empty fuzzy subset on M. Then f is an l-prime fuzzy suboperand of Q if and only if $\inf_{p\in P} f(q\theta(p\otimes q')) = \inf_{p\in P} f(\phi(q\otimes p)q') = f(q) \vee f(q')$ for all $q, q' \in Q$.

It is relevant to mention here that the lattices under consideration in the following theorem are actually lattices because they are sublattices of the lattices considered in Corollary 3.2.

Theorem 3.6. *Let S and T be semigroups with weak local units. If S and T are strongly Morita equivalent via the Morita context $< S, T, {}_SP_T, {}_TQ_S, \theta, \phi >$, then the following lattices are isomorphic:*

1. *the lattice of all l-prime fuzzy ideals of S;*
2. *the lattice of all l-prime fuzzy ideals of T;*
3. *the lattice of all l-prime fuzzy suboperands of ${}_SP_T$;*
4. *the lattice of all l-prime fuzzy suboperands of ${}_TQ_S$.*

Proof. (1) $\simeq$ (3) In view of Corollary 3.2 and the fact that an l-prime fuzzy ideal is a fuzzy ideal it is sufficient to prove that for an l-prime fuzzy ideal μ of S, μ^+ is an l-prime fuzzy suboperand of P and for an l-prime fuzzy suboperand f of P, $f^{+'}$ is an l-prime fuzzy ideal of S.

Let μ be an l-prime fuzzy ideal of S and $p, p' \in P$. Then

$$\begin{aligned}
\inf_{q\in Q} \mu^+(\theta(p\otimes q)p') &= \inf_{q,q'\in Q} \mu(\theta(\theta(p\otimes q)p'\otimes q')) \\
&= \inf_{q,q'\in Q} \mu(\theta(p\otimes q)\theta(p'\otimes q')) \\
&= \inf_{q,q'\in Q} \mu(\theta(p\otimes q)) \vee \mu(\theta(p'\otimes q')) \\
&= \inf_{q\in Q} \mu(\theta(p\otimes q)) \vee \inf_{q'\in Q} \mu(\theta(p'\otimes q')) \\
&= \mu^+(p) \vee \mu^+(p').
\end{aligned}$$

Hence μ^+ is an l-prime fuzzy suboperand of P (*cf.* Theorem 3.4). Now let f be an l-prime fuzzy suboperand of P and $x, y \in S$. Then

$$\begin{aligned}
f^{+'}(xy) &= \inf_{p\in P} f(xyp) \\
&= \inf_{p\in P} \Big(\big(\inf_{p'\in P} f(xp')\big) \vee f(yp)\Big) \\
&= \inf_{p'\in P} f(xp') \vee \inf_{p\in P} f(yp) \\
&= f^{+'}(x) \vee f^{+'}(y).
\end{aligned}$$

Hence $f^{+'}$ is an l-prime fuzzy ideal of S.

The proofs of (2) $\simeq$ (3), (1) $\simeq$ (4) and (2) $\simeq$ (4) follow by relevant modifications of the arguments applied above. ■

Now we use two more fuzzy notions viz. the notions of fuzzy congruences (*cf.* Definition 2.9) of a semigroup and fuzzy operand equivalences (Definition 3.7 follows) of a bioperand and obtain four more lattices associated respectively with each component of a Morita context of strongly Morita equivalent semigroups.

Definition 3.7. *Let S and T be two semigroups and M a bioperand over S and T. A fuzzy equivalence relation σ on M is called a fuzzy operand equivalence on M if $\sigma(sx, sy) \geq \sigma(x, y)$ and $\sigma(xt, yt) \geq \sigma(x, y)$ for all $x, y \in M$, for all $s \in S$ and for all $t \in T$.*

We note that the set of all fuzzy congruences of a semigroup S and the set of all fuzzy operand equivalences of a bioperand ${}_SP_T$ both form lattices with respect to meet ($\wedge$) and join ($\vee$) defined by $\mu \wedge \nu = \mu \cap \nu$ and $\mu \vee \nu = \sup_{n\in\mathbb{N}}(\mu \circ \nu)^n$, for any two fuzzy relations μ and ν.

In the following theorem we show that the lattice of fuzzy congruences and that of fuzzy operand equivalences related to a Morita context are isomorphic.

Theorem 3.8. *Let S and T be semigroups with common joint weak local* *units. If S and T are strongly Morita equivalent via the Morita context $< S, T, {}_SP_T, {}_TQ_S, \theta, \phi >$, then the following lattices are isomorphic:*

(1) *the lattice $FCON(S)$ of fuzzy congruences of S;*
(2) *the lattice $FCON(T)$ of fuzzy congruences of T;*
(3) *the lattice $FOE(P)$ of fuzzy operand equivalences of ${}_SP_T$;*
(4) *the lattice $FOE(Q)$ of fuzzy operand equivalences of ${}_TQ_S$.*

Proof. We first show the isomorphism between (1) and (3).

For a fuzzy congruence ρ of S, we define a fuzzy relation $\rho^+ : P \times P \to [0,1]$ by $\rho^+(p, p') := \inf_{q \in Q} \rho(\theta(p \otimes q), \theta(p' \otimes q))$, and for a fuzzy operand equivalence σ of P, define $\sigma^{+'} : S \times S \to [0,1]$ by $\sigma^{+'}(s, s') := \inf_{p \in P} \sigma(sp, s'p)$.

Let $s \in S,\ p, p' \in P$. Then

$$\begin{aligned}\rho^+(sp, sp') &= \inf_{q \in Q} \rho(\theta(sp \otimes q), \theta(sp' \otimes q)) \\ &= \inf_{q \in Q} \rho(s\theta(p \otimes q), s\theta(p' \otimes q)) \\ &\geq \inf_{q \in Q} \rho(\theta(p \otimes tq), \theta(p' \otimes tq)) \ \ \text{(as } \rho \text{ is a fuzzy congruence of } S) \\ &= \rho^+(p, p').\end{aligned}$$

Again let $p, p' \in P,\ t \in T$. Then

$$\begin{aligned}\rho^+(pt, p't) &= \inf_{q \in Q} \rho(\theta(pt \otimes q), \theta(p't \otimes q)) \\ &= \inf_{q \in Q} \rho(\theta(p \otimes tq), \theta(p' \otimes tq)) \\ &\geq \inf_{q \in Q} \rho(\theta(p \otimes q), \theta(p' \otimes q)) \quad = \rho^+(p, p').\end{aligned}$$

So ρ^+ is a fuzzy operand equivalence of ${}_SP_T$.

Now let $x, y, x', y' \in S$. Then

$$\begin{aligned}\sigma^{+'}(xy, xy') &= \inf_{p \in P} \sigma(xyp, xy'p) \\ &\geq \inf_{p \in P} \sigma(yp, y'p) \qquad \text{(as } \sigma \text{ is fuzzy operand equivalence of } {}_SP_T) \\ &= \sigma^{+'}(y, y').\end{aligned}$$

Also

$$\begin{aligned}\sigma^{+'}(xy, x'y) &= \inf_{p \in P} \sigma(xyp, x'yp) \\ &\geq \inf_{p \in P} \sigma(xp, x'p) \quad = \sigma^{+'}(x, x').\end{aligned}$$

So $\sigma^{+'}$ is a fuzzy congruence of S. Now let us define $\Pi^+ : FCON(S) \to FOE(P)$ by

$$\Pi^+(\rho) := \rho^+$$

and define $\Pi^{+'} : FOE(P) \to FCON(S)$ by

$$\Pi^{+'}(\sigma) := \sigma^{+'}.$$

Then for $s, s' \in S$, we have

$$\begin{aligned}(\Pi^{+'} \circ \Pi^{+})(\rho)(s,s') &= (\rho^{+})^{+'}(s,s') \\ &= \inf_{p\in P} \rho^{+}(sp, s'p) \\ &= \inf_{p\in P} \inf_{q\in Q} \rho(\theta(sp\otimes q), \theta(s'p\otimes q)) \\ &= \inf_{p\in P} \inf_{q\in Q} \rho(s\theta(p\otimes q), s'\theta(p\otimes q)) = \rho(s,s').\end{aligned}$$

The last equality holds since ρ is a fuzzy congruence of S and p, q can be chosen such that $\theta(p\otimes q)$ is a common joint weak right local unit of s and s'.

Again for $p, p'' \in P$, we have

$$\begin{aligned}(\Pi^{+} \circ \Pi^{+'})(\sigma)(p,p'') &= (\sigma^{+'})^{+}(p,p'') \\ &= \inf_{q\in Q} \sigma^{+'}(\theta(p\otimes q), \theta(p''\otimes q)) \\ &= \inf_{q\in Q} \inf_{p'\in P} \sigma(\theta(p\otimes q)p', \theta(p''\otimes q)p') \\ &= \inf_{q\in Q} \inf_{p'\in P} \sigma(p\phi(q\otimes p'), p''\phi(q\otimes p')) = \sigma(p,p'').\end{aligned}$$

The last equality holds since σ is a fuzzy operand equivalence of P and p', q can be chosen such that $p\phi(q\otimes p') = p$ and $p''\phi(q\otimes p') = p''$. Hence Π^{+} and $\Pi^{+'}$ are mutually inverse bijections.

Now let ρ_1, ρ_2 be two fuzzy congruences of S. Then $\rho_1 \subseteq \rho_2$ implies

$$\begin{aligned}\rho_1^{+}(p,p') &= \inf_{q\in Q} \rho_1(\theta(p\otimes q), \theta(p'\otimes q)) \\ &\leq \inf_{q\in Q} \rho_2(\theta(p\otimes q), \theta(p'\otimes q)) = \rho_2^{+}(p,p')\end{aligned}$$

for any $p, p' \in P$.

Hence Π^{+} is a lattice isomorphism (cf. [4, Proposition 2.19]).

In order to complete the proof we only prescribe below the functions which give rise to the relevant isomorphisms. We omit the details as they are the relevant modifications of the arguments applied above.

For a fuzzy congruence ρ of T we define $\Pi^{*} : FCON(T) \to FOE(P)$ by

$$\Pi^{*}(\rho) := \rho^{*}$$

and for a fuzzy operand equivalence σ of P we define $\Pi^{*'} : FOE(P) \to FCON(T)$ by

$$\Pi^{*'}(\sigma) := \sigma^{*'},$$

where $\rho^{*} : P \times P \to [0,1]$ is defined by

$$\rho^{*}(p,p') := \inf_{q\in Q} \rho(\phi(q\otimes p), \phi(q\otimes p')),$$

and $\sigma^{*'} : T \times T \to [0,1]$ is defined by

$$\sigma^{*'}(t,t') := \inf_{p\in P} \sigma(pt, pt').$$

For a fuzzy congruence ρ of S we define ${}^{+}\Pi : FCON(S) \to FOE(Q)$ by

$$^{+}\Pi(\rho) :=^{+} \rho,$$

and for a fuzzy operand equivalence σ of Q we define ${}^{+'}\Pi : FOE(Q) \to FCON(S)$ by

$$^{+'}\Pi(\sigma) :=^{+'} \sigma,$$

where ${}^{+}\rho : Q \times Q \to [0,1]$ is defined by

$$^{+}\rho(q,q') := \inf_{p \in P} \rho(\theta(p \otimes q), \theta(p \otimes q')),$$

and ${}^{+'}\sigma : S \times S \to [0,1]$ is defined by

$$^{+'}\sigma(s,s') := \inf_{q \in Q} \sigma(qs, qs').$$

For a fuzzy ideal ρ of T we define ${}^{*}\Pi : FCON(T) \to FOE(Q)$ by

$$^{*}\Pi(\rho) :=^{*} \rho,$$

and for a fuzzy operand equivalence σ of Q we define ${}^{*'}\Pi : FOE(Q) \to FCON(T)$ by

$$^{*'}\Pi(\sigma) :=^{*'} \sigma,$$

where ${}^{*}\rho : Q \times Q \to [0,1]$ is defined by

$$^{*}\rho(q,q') := \inf_{p \in P} \rho(\phi(q \otimes p), \phi(q' \otimes p)),$$

and ${}^{*'}\sigma : T \times T \to [0,1]$ is defined by ${}^{*'}\sigma(t,t') := \inf_{q \in Q} \sigma(tq, t'q)$. ■

In order to conclude the paper using the notions of fuzzy semilattice congruences of a semigroup and fuzzy operand equivalences of a bioperand we obtain following four lattices and establish their isomorphism in Theorem 3.10.

Definition 3.9. [20] *Let S be a semigroup. A fuzzy equivalence relation μ on S is called a fuzzy semilattice congruence if*

(1) *μ is a fuzzy congruence (i.e., $\mu(a,b) \le \mu(ac,bc)$, $\mu(a,b) \le \mu(ca,cb)$ for all $a,b,c \in S$),*
(2) *$\mu(a^2,a) = 1$ and $\mu(ab,ba) = 1$ for all $a,b \in S$.*

The set of all fuzzy semilattice congruences on S are generally denoted by $FSC(S)$.

It is a matter of routine verification that for a semigroup S the set of all fuzzy semilattice congruences of S forms a lattice as it becomes a sublattice of the lattice of fuzzy congruences of S. Also for any Morita context

$< S, T, {}_SP_T, {}_TQ_S, \theta, \phi >$, $V = \{\sigma |\ \sigma \in FOE(P),\ \sigma(\theta(p \otimes q)p, p) = 1,\ \sigma(\theta(p \otimes q)p', \theta(p' \otimes q)p) = 1\}$ and $W = \{\sigma |\ \sigma \in FOE(Q),\ \sigma(q\theta(p \otimes q), q) = 1,\ \sigma(q'\theta(p \otimes q), q\theta(p \otimes q')) = 1\}$ become sublattices of the lattices of fuzzy operand equivalences of P and that of Q, respectively. We deduce in the following theorem that any two of the four lattices, associated with the four components of a Morita context of strongly Morita equivalent semigroups with common join weak local units, are isomorphic.

Theorem 3.10. *Let S and T be semigroups with common joint weak local units. If S and T are strongly Morita equivalent via the Morita context $< S, T, {}_SP_T, {}_TQ_S, \theta, \phi >$, then the following lattices are isomorphic:*

(1) *the lattice $FSC(S)$ of all fuzzy semilattice congruences of S;*
(2) *the lattice $FSC(T)$ of all fuzzy semilattice congruences of T;*
(3) *the lattice $V = \{\sigma |\ \sigma \in FOE(P),\ \sigma(\theta(p \otimes q)p, p) = 1,\ \sigma(\theta(p \otimes q)p', \theta(p' \otimes q)p) = 1\}$;*
(4) *the lattice $W = \{\sigma |\ \sigma \in FOE(Q),\ \sigma(q\theta(p \otimes q), q) = 1,\ \sigma(q'\theta(p \otimes q), q\theta(p \otimes q')) = 1\}$.*

Proof. We first show the isomporphism between (1) and (3).

For a fuzzy semilattice congruence ρ of S, we define a fuzzy relation $\rho^+ : P \times P \to [0, 1]$ by

$$\rho^+(p, p') := \inf_{q \in Q} \rho(\theta(p \otimes q), \theta(p' \otimes q)),$$

and for $\sigma \in V$, we define $\sigma^{+'} : S \times S \to [0, 1]$ by

$$\sigma^{+'}(s, s') := \inf_{p \in P} \sigma(sp, s'p).$$

Then we know that ρ^+ is a fuzzy operand equivalence of ${}_SP_T$ (*cf.* the proof of Theorem 3.8). Now let $p,\ p' \in P$ and $q \in Q$. Then

$$\begin{aligned} \rho^+(\theta(p \otimes q)p, p) &= \inf_{q \in Q} \rho(\theta(\theta(p \otimes q)p \otimes q), \theta(p \otimes q)) \\ &= \inf_{q \in Q} \rho(\theta(p \otimes q)\theta(p \otimes q), \theta(p \otimes q)) \\ &= 1 \text{ (since } \rho \text{ is a fuzzy semilattice congruence on } S) \end{aligned}$$

and

$$\begin{aligned} \rho^+(\theta(p \otimes q)p', \theta(p' \otimes q)p) &= \inf_{q \in Q} \rho(\theta(\theta(p \otimes q)p' \otimes q), \theta(\theta(p' \otimes q)p \otimes q)) \\ &= \inf_{q \in Q} \rho(\theta(p \otimes q)\theta(p' \otimes q), \theta(p' \otimes q)\theta(p \otimes q)) \\ &= 1 \text{ (since } \rho \text{ is a fuzzy semilattice congruence on } S). \end{aligned}$$

Hence $\rho^+ \in V$. Also we know that $\sigma^{+'}$ is a fuzzy congruence on S (*cf.* the proof of Theorem 3.8). Let $a, b \in S$. Then

$$\begin{aligned}\sigma^{+'}(a^2, a) &= \inf_{p\in P} \sigma(a^2p, ap)\\ &= \inf_{p\in P} \sigma(\theta(p'\otimes q')\theta(p'\otimes q')p, \theta(p'\otimes q')p) \text{ (since } \theta \text{ is surjective)}\\ &= \inf_{p\in P} \sigma(\theta(p'\otimes q')p'\phi(q'\otimes p), p'\phi(q'\otimes p))\\ &\geq \sigma(\theta(p'\otimes q')p', p') \text{ (since } \sigma \in V)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\sigma^{+'}(ab, ba) &= \inf_{p\in P} \sigma(abp, bap)\\ &= \inf_{p\in P} \sigma(\theta(p'\otimes q')\theta(p''\otimes q'')p, \theta(p''\otimes q'')\theta(p'\otimes q')p) \text{ (since } \theta \text{ is surjective)}\\ &\geq \inf_{p\in P} \{\sigma(\theta(p'\otimes q')\theta(p''\otimes q'')p, \theta(p''\otimes q'')\theta(p\otimes q')p')\\ &\quad \wedge\sigma(\theta(p''\otimes q'')\theta(p\otimes q')p', \theta(p''\otimes q'')\theta(p'\otimes q')p)\} \text{ (since } \sigma\circ\sigma\subseteq\sigma)\\ &\geq \inf_{p\in P} \{\sigma(\theta(p'\otimes q')(\theta(p''\otimes q'')p), \theta((\theta(p''\otimes q'')p)\otimes q')p')\\ &\quad \wedge\sigma(\theta(p\otimes q')p', \theta(p'\otimes q')p)\} \text{ (since } \sigma \in V \text{ and } S \text{ is a monoid)}\\ &= 1 \text{ (since } \sigma \in V).\end{aligned}$$

So $\sigma^{+'}$ is a fuzzy semilattice congruence on S. Now let us define $\Pi^+ : FSC(S) \to V$ by

$$\Pi^+(\rho) := \rho^+$$

and define $\Pi^{+'} : V \to FSC(S)$ by

$$\Pi^{+'}(\sigma) := \sigma^{+'}.$$

In view of discussion made before the theorem it follows from the proof of Theorem 3.8 that Π^+ is a lattice isomorphism with $\Pi^{+'}$ as its inverse.

The other isomorphisms follow by relevant modifications of the arguments applied above. ∎

4. Concluding Remark

If we look at the main four results on isomorphism of the lattices (see Cor. 3.2 and Theorems 3.6, 3.8 and 3.10) related with the different components of Morita context of strongly equivalent semigroups from a different point of view we can conclude that the fuzzy notions, used to obtain the said lattices, are in fact Morita invariants (*i.e.,* which remains invariant on passing from one component to the other components of a Morita context) for semigroup content.

For further study it would be interesting to find not only some other notions which are Morita invariants but also some notions which are not.

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