B.Sc./2nd Sem/MTM/24(NEP)

2024

2nd Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MJ 2-T (Single Core Major)

[Algebra]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any ***ten*** of the following questions :

2×10=20

1. Two complex numbers z_1, z_2 are such that $|z_1+z_2|=|z_1-z_2|$. Prove that principal amplitude of z_1 and z_2 differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

2. State the Descartes' rule of sign.

3. Is the function $f(x)=x^2,\ \forall x\in[-1,1]$ injective? Justify

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your answer. Is it possible to set a new domain so that f can be injective there on?

4. Find the greatest value of a^2b^3 where a, b are positive real numbers satisfying $3a + 4b = 5$. Find the values of a and b also.

5. Let $f: A \to B$ and $g: B \to C$ be two maps such that $g \circ f: A \to C$ is surjective. Prove that g is surjective.

6. Find the remainder when 12^{201} is divided by 7.

7. If a, b, c be positive real numbers, not all equal, prove that

$$\left(a^3 + b^3 + c^3\right)\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) > 9$$

8. Say True or False with justification :

"Let $f: A \to B$ and $h, t: B \to C$ be mappings. Then $h \circ f = t \circ f$ implies $h = t$."

9. By Euclidean algorithm, find integers u and v such that $52u - 91v = 78$.

10. Let A be a real matrix of $m \times n$ order. Is the map $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(v) = Av$ a linear map? Why?

11. Let exp (z) denote the complex exponent value of the complex number z. Prove that $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for complex numbers z_1 and z_2.

12. Find a basis for the set of vectors in $\mathbb{R}^3$ lying in the plane $x + 2y + z = 0$.

13. Find A^{-1} by using elementary row operations, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}.$$

14. Find the special roots of the equation $x^{12} - 1 = 0$.

15. Let A be a square real matrix of order 7, with characteristic polynomial $\lambda^7 - 2\lambda^6 - 20\lambda^5 - 24\lambda^4$. Find the eigen values of A and their multiplicity (algebraic).

Group - B

Answer any ***four*** of the following questions : 5×4=20

16. Define nth root of a non-zero complex number where $n \in \mathbb{N}$. Prove that $\left(z^{\frac{1}{n}}\right)^m = \left(z^m\right)^{\frac{1}{n}}$, where $z \in \mathbb{C}^*$ and $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$. Hence show that $z^{p/q} = z^{m/n}$ where $p, q \in \mathbb{N}$ such that $pn = mq$. 1+3+1

17. If α, β, γ be the roots of the equation $x^3 + 3x + 1 = 0$, find the equation whose roots are $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}$, $\frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$.

18. A relation β is defined on $\mathbb{Z}$ by "$x\beta y$ if and only if $x^2 - y^2$ is multiple of 5" for $x, y \in \mathbb{Z}$. Prove that β is equivalence relation on $\mathbb{Z}$. Show that there are three distinct equivalence classes.

19. State the Euler's theorem on congruence. Prove that $m^{\phi(n)} + n^{\phi(m)} \equiv 1(\bmod mn)$ where m, n are natural numbers and prime to each other and ϕ is the Euler's Phi function. 1+4

20. For what values of k, the planes $x + y + 1 = 0$, $4x + y - z = k$ and $5x - y - 2z = k^2$ form a triangular prism?

21. If a, b, c, d be all positive real numbers and $s = a + b + c + d$, prove that

$$81abcd \le (s-a)(s-b)(s-c)(s-d) \le \frac{81}{256}s^4.$$

Group - C

Answer any ***two*** of the following questions :

10×2=20

22. (i) Prove that $\sin\left[i\log\frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2+b^2}$.

(ii) If the equation $x^4 + px^3 + qx^2 + rx + s = 0$ has roots of the form $\alpha \pm i\alpha$, $\beta \pm i\beta$ where α and β are real, prove that $p^2 - 2q = 0$ and $r^2 - 2qs = 0$.

(iii) Let ρ be an equivalence relation on a set S and $a, b \in S$. Then prove that $cl(a) = cl(b)$ if and only if $a\rho b$. 3+4+3

23. (i) Use theory of congruences to show that $43 \mid 6^{n+2} + 7^{2n+1}$ for all $n \in \mathbb{N}$.

(ii) Use Cayley-Hamilton theorem to find A^{100} where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(iii) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $\Sigma(\beta + \gamma - \alpha)^3$. 3+3+4

24. (i) Find all vectors v in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $x \mapsto Ax$, $\forall x \in \mathbb{R}^4$, for the given matrix

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}.$$ Is the vector $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ in the range of the above transformation? Why or why not?

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(ii) Define eigen vectors of a square matrix A over a field F. Prove that two eigen vectors of A corresponding to two different eigen values are linearly independent. (3+3)+(1+3)

25. (i) Let A be a 3×3 real matrix having eigen values 5, 2, 2. The eigen vectors of A corresponding to eigen values 5 and 2 are $c\begin{pmatrix}1\\1\\1\end{pmatrix}$, $c\neq 0$ and $a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\end{pmatrix}$, $(a, b)\neq(0, 0)$ respectively. Find the matrix A.

(ii) Find all real λ for which the rank of the matrix $\begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & -1 & \lambda\\5 & 7 & 1 & \lambda^2\end{bmatrix}$ is 2.

(iii) If $\gcd(a, b) = 1$, then show that $\gcd\left(a^2-b^2, a^2+b^2\right)=1$ or 2. 4+3+3