B.Sc/1st Sem/MATH/23(NEP)

2023

1st Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MJ 1-T (Single Core Major)

(Calculus, Geometry and Ordinary Differential Equation)

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any *ten* questions where symbols have their usual meanings : 2×10=20

1. Prove that $y = ax + b$ is an asymptote to the curve $y = ax + b + \frac{\sin x}{x}$.

2. For a curve, if $x\sin\theta + y\cos\theta = f'(\theta)$ and $x\cos\theta - y\sin\theta = f''(\theta)$, prove that $s = f(\theta) + f''(\theta) + k$, where k is a constant.

P.T.O.

3. Find the volume of the solid generated by the revolution of the area enclosed by the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

4. If m, n being positive integers greater than 1, derive the following reduction formula :

$$I_{m,n} = \int (\sin x)^m (\cos x)^n \, dx = \frac{(\sin x)^{m+1} (\cos x)^{n-1}}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}.$$

5. Find the values of b and g such that the equation $4x^2 + 8xy + by^2 + 2gx + 4y + 1 = 0$ represents a conic without any centre.

6. Find the equation of the sphere which passes through the origin and touches the sphere $x^2 + y^2 + z^2 = 56$ at the point (2, –4, 6).

7. Show that the general solution of the equation $\frac{dy}{dx} + Py = Q$ can be written in the form $y = k(u - v) + v$, where k is a constant and u and v are its two particular solutions.

8. If $y = \frac{x^3}{x^2 - 1}$, then find y_n.

9. Find the range of values of x for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upward.

10. Find the envelops of lines $\frac{x}{a}+\frac{y}{b}=1$, where $ab = c^2$.

11. Find the point of contact of the plane $8x-6y-z=5$ and the paraboloid $3x^2-2y^2=6z$.

12. Find the equation of the sphere of which the circle $xy+yz+zx=0$, $x+y+z=3$ is a great circle.

13. Verify that $\lim_{x\to\infty}\left\{x-x^2\log\left(1+\frac{1}{x}\right)\right\}=\frac{1}{2}$.

14. Find the length of the curve $x=e^{\theta}\sin\theta$, $y=e^{\theta}\cos\theta$ between $\theta=0$ to $\frac{\pi}{2}$.

15. Find the values of the constant λ such that $\left(2xe^{y}+3y^{2}\right)\frac{dy}{dx}+\left(3x^{2}+\lambda e^{y}\right)=0$ is exact.

Group - B

Answer any ***four*** questions where symbols have their usual meanings : 5×4=20

16. The circle $x^2+y^2=a^2$ is divided by the hyperbola $x^2-2y^2=\frac{a^2}{4}$. Find out the area of the portion of the circle which does not contained in the hyperbola.

P.T.O.

17. Establish the nature of the following curve and find the centre, semi-axes and eccentricity, if any :

$$x^2+24xy-6y^2+28x+36y+16=0.$$

18. If the tangents at P and Q of a parabola meet at T and S be the focus of the parabola, then prove that, $ST^2 = SP.SQ$.

19. Reduce the equation $x^2p^2+yp(2x+y)+y^2=0$ to Clairaut's form with the substitution of $y=u$, $xy=v$ and obtain complete primitive.

20. If $I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m x\cos^n x\,dx$, m, n being positive integers greater than 1, prove that $I_{m,n}=\frac{n-1}{m+n}I_{m,n-2}$.

Hence find the value of $\int_0^1 x^6\sqrt{1-x^2}\,dx$.

21. Show that the centre of the sphere which always touch the $y=mx$, $z=c$ and $y=-mx$, $z=-c$ lie on the surface $mxy+cz\left(1+m^2\right)=0$.

Group - C

Answer any *two* questions where symbols have their usual meanings : 10×2=20

22. (i) Determine the asymptotes of the following curve :

$$x=\frac{1}{t^4-1},\ y=\frac{t^3}{t^4-1}.$$ 3

(ii) Trace the curve $x^2y^2 = 9\left(y^2 - x^2\right)$. 4

(iii) Find the evolute of the curve $x = at^2$, $y = 2at$ (evolute is the envelope of normals). 3

23. (i) If $2r$ be the distance between two parallel tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$, then show that $\left(a^2-r^2\right)l^2+\left(b^2-r^2\right)m^2+\left(c^2-r^2\right)n^2=0$.

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(ii) Show that the straight lines in which the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2=0$ are perpendicular if $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$ and parallel if $bcu^2+cav^2+abw^2=0$. 5

24. (i) Show that the feet of the normal from the point (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the intersection of the ellipsoid and the cone

$$\frac{\alpha a^2\left(b^2-c^2\right)}{x}+\frac{\beta b^2\left(c^2-a^2\right)}{y}+\frac{\gamma c^2\left(a^2-b^2\right)}{z}=0.$$

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P.T.O.

(ii) Solve $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2$. 5

25. (i) Prove that the locus of the middle point of any focal chord of the conic $\frac{l}{r}=1+e\cos\theta$ is $r\left(1-e^2\cos^2\theta\right)=el\cos\theta$. 5

(ii) If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardiode $r=a(1+\cos\theta)$, which passes through the pole, then prove that $\rho_1^2+\rho_2^2=\frac{16}{9}a^2$. 5