B.Sc./2nd Sem (H)/MTM/24(CBCS)

# 2024

## 2nd Semester Examination

## MATHEMATICS (Honours)

### Paper : GE 2-T

### [Algebra]

### [CBCS]

Full Marks : 60 Time : Three Hours

*The figures in the margin indicate full marks.*
*Candidates are required to give their answers in their own words as far as practicable.*

1. Answer any ***ten*** questions : $2\times10=20$

(a) Find the principal value of the complex number $z=-1-i$.

(b) Show that the equation $x^3-3x^2-9x+27=0$ has a multiple root.

(c) Use Descartes's rule of sign to show that the equation $x^4-x+3=0$ has no real root.

(d) Show that if a function $f\circ g$ is injective and $g$ is surjective function, then $f$ is an injective function.

(e) Find two integers $u$ and $v$ satisfying $54u+24v=30$.

P.T.O.

(f) If $x$ be real, prove that $i\log\frac{1+ix}{1-ix}=-2\tan^{-1}x$.

(g) Prove that the product of any three consecutive integers is divisible by 6.

(h) Find the value of $k$ for which the set $S=\{(k,1,1),(1,k,1),(1,1,k)\}$ is linearly independent.

(i) Do the polynomials $x^3-2x^2+1$, $4x^3-x+3$ and $3x-2$ generate $P_3(R)$? Justify.

(j) Let $P$ be an orthogonal matrix with $\det(P)=-1$. Prove that $-1$ is an eigen value of $P$.

(k) Find the greatest value of $b^3a^2$ where $a, b$ are positive real number such that $a+b=10$. Determine the values of $a, b$ for which the greatest value is attained.

(l) Using Euler's theorem, find the unit digit of $3^{100}$.

(m) Find the rank of the matrix $A=\begin{bmatrix}0 & 0 & 2\\ 1 & 3 & 2\\ 2 & 6 & 2\end{bmatrix}$.

(n) Express the matrix $A^{-1}$ in terms of $A$, where

$$A=\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{bmatrix}.$$

(o) If $a, b, c, d$ be positive real numbers, not all equal, prove that $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)>16.$

2. Answer any ***four*** questions : 5×4=20

(a) Solve the equation by Cardan's method :

$$x^3 - 15x^2 - 33x + 847 = 0.$$

(b) Find the characteristic equation of the matrix

$$A=\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}.$$ Hence find $A^{-1}$.

(c) Find the dimension of the subspace $S$ of $R^3$ defined by :

$$S=\{(x,y,z)\in R^3 : x+2y=z;\ 2x+3z=y\}.$$

(d) If $(2+\sqrt{3})^n = I+f$ where $I$ and $n$ are positive integers and $0<f<1$, show that $I$ is an odd integer and $(1-f)(I+f)=1$.

(e) If $\alpha, \beta, \gamma$ be the roots of the equation $x^3+3x+1=0$, find the equation whose roots are $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}, \frac{\beta}{\gamma}+\frac{\gamma}{\beta}, \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}.$

P.T.O.

(f) Show that the map $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by,

$T(x,y,z)=(x+y+z,\ 2x+y+2z,\ x+2y+z)$,

$\forall x,\ y,\ z \in \mathbb{R}$ is linear. Find the matrix representation $[T]_B^{B'}$ of $T$ with respect to the ordered bases $B=\{(1,0,0),(1,1,0),(1,1,1)\}$ and $B'=\{(1,1,0),(1,0,1),(0,1,1)\}$. 2+3

3. Answer any ***two*** questions : 10×2=20

(a) (i) Show that $\gcd(a+2,\ a)=1$ or 2 for every integer $a$.

(ii) Verify whether the set $\{3x-1,\ x^3+1,\ x-3\}$ is a basis for the vector space $P_3(\mathbb{R})$.

(iii) Prove that $n^4 + 4^n$ is a composite number for all $n > 1$. 3+3+4

(b) (i) Use principle of mathematical induction and establish the formula for all integers $n \geq 1$,

$$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{4n^3-n}{3}.$$

(ii) Show that the eigen vectors corresponding to the distinct eigen values are linearly independent. 6+4

(c) (i) Using Cayley-Hamilton theorem for the matrix $A$, compute $A^{50}$.

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

(ii) Show that the eigen value of an orthogonal matrix has a unit modulus. 5+5

(d) (i) If $d=\gcd(a,m)$, prove that

$$ax\equiv ay \pmod{m} \Leftrightarrow x\equiv y\left(\bmod \frac{m}{d}\right).$$

(ii) Solve by Ferrari's method :

$x^4+4x^3-6x^2+20x+8=0.$ 4+6

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