B.Sc./5th Sem (H)/MATH/22(CBCS)

2022

5th Semester Examination

MATHEMATICS (Honours)

Paper : C 12-T

[Group Theory II]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Attempt any ***ten*** questions : $2\times10=20$

(a) Find two non-isomorphic groups H_1 and H_2 such that $Aut(H_1)$ is isomorphic with $Aut(H_2)$.

(b) Let G be a group. Then prove that $|Inn(G)| = 1$ if and only if G is commutative.

(c) Verify whether $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_4$.

(d) Let G be a cyclic group of order 2023. Find the number of automorphisms defined on G.

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(e) Express $U(165)$ as an external direct product of cyclic groups of the form $\mathbb{Z}_n$.

(f) Give an example of a group G such that $|G| = 12$ and G has more than one subgroup of order 6.

(g) Define characteristic subgroup of a group G. Is it true that every normal subgroup is characteristic? Give reasons in support of your answer.

(h) Find the class equation for the Klein's four group.

(i) Let p, q be odd primes and let m and n be positive integers. Is $U(p^m) \times U(q^n)$ cyclic? Justify your answer. Here $U(n)$ denotes the group of units modulo n.

(j) Let G be a p-group (where p is a prime) and H be a non-trivial homomorphic image of G. Then prove that H is also a p-group.

(k) Let R denote the set of all polynomials with integer coefficients in the independent variables x_1, x_2, x_3. Let S_3 act on R by $\sigma \cdot p(x_1, x_2, x_3) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. Find the stabilizer of the polynomial $x_1 x_2$ under the action of G.

(l) Express the Klein's four group as an internal direct product of two of its proper subgroups.

(m) Find the conjugacy classes of $cl((1,2))$ and $cl((1,2,3))$ in S_3.

(n) Verify whether a non-commutative group of order 343 is simple.

(o) State fundamental theorem for finite abelian groups.

Group - B

2. Attempt any *four* questions : 5×4=20

(a) Prove that commutator subgroup G' of a group G is a characteristic subgroup of G. 5

(b) Let G be a group. Define commutator subgroup of G. Prove that Commutator subgroup G' is a normal subgroup of G and G/G' is commutative. 1+4

(c) Find all subgroups of order 3 in $\mathbb{Z}_9 \oplus \mathbb{Z}_3$. 5

(d) Determine all non-isomorphic abelian groups of order 720. 5

(e) Let G be a group of order 60. If Sylow 3-subgroup is normal in G then show that Sylow 5-subgroup is also normal in G. 5

(f) Let G be a group. Prove that the mapping $\phi : G \times G \to G$ defined by $\phi(g,a) = g \cdot a = gag^{-1}$ is a group action. Find its kernel and stabilizer G_a. 5

P.T.O.

Group - C

3. Attempt any *two* questions : 10×2=20

(a) (i) Find $Aut(\mathbb{Z})$.

(ii) If G is a non-abelian group then show that $Aut(G)$ can not be cyclic.

(iii) Prove that $Inn(G) \approx \frac{G}{Z(G)}$, where $Inn(G)$ is the group of inner automorphism of G and $Z(G)$ is the centre of G. 2+3+5

(b) (i) Show that $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

(ii) Find all conjugacy classes of the Dihedral group D_8 of order 8 and hence verify the class equation.

(iii) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 2+3+5

(c) (i) State Cauchy's theorem. Use Cauchy's theorem to prove that if a finite group G is a p-group then $|G| = p^n$ for some positive integer n, where p is a prime.

(ii) Let G be a group of order pn, where p is a prime and $p > n$. Show that there exists a subgroup of order p in G which is normal.

(iii) Find the number of elements of order 5 in the direct product $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$. 3+3+4

(d) (i) Let G be a group acting on a non-empty set S and $a \in S$. Then prove that $|[a]| = [G : G_a]$ where $[a]$ denotes the orbit of a and G_a denotes the stabilizer of a.

(ii) Let G be a finite group and H be a proper subgroup of G with index n such that $|G|$ does not divide $n!$. Using group action show that G contains a non-trivial normal subgroup. Hence show that a simple group of order 63 cannot contain a subgroup of order 21.

4+(3+3)
