B.Sc./5th Sem (H)/MATH/23(CBCS)

2023

5th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 1-T

[CBCS]

Full Marks : 60 — Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Linear Programming]

1. Answer any ***ten*** questions : 2×10=20

   (a) Give three limitations of a linear programming problem.

   (b) Find the possible number of basic solutions in a system of m equations in n unknowns.

   (c) Show that $X = \{x : |x| \leq 2\}$ is a convex set.

   (d) Prove that dual of the dual is the primal.

   (e) Prove that the solution of the transportation problem has always a solution.

P.T.O.

(f) Write the basic difference of transportation and assignment problems.

(g) Show that whatever may be the value of a, the game with the following payoff matrix is strictly determinable :

		B: I	B: II
A	I	3	7
	II	-3	a

(h) In the following equations find the basic solution with x_3 as the non-basic solution

$$x_1 + 4x_2 - x_3 = 3$$
$$5x_1 + 2x_2 + 3x_3 = 4.$$

(i) Extreme points of a convex set are its boundary points. Is the converse true? Justify.

(j) What is simplex? Give an example of a simplex in 3-dimension.

(k) State fundamental theorem of an L.P.P.

(l) State the weak duality theorem of an L.P.P.

(m) Distinguish between a regular simplex method and a dual simplex method.

(n) What do you mean by two-person zero sum game and its value?

(o) Prove that the number of basic variables in a transportation problem with m origins and n destinations is atmost $m + n - 1$.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Show that if any variable of the primal problem be unrestricted in sign, then the corresponding constraint of the dual will be an equality.

(b) Solve the following L.P.P.

$$\text{Max } Z = 2x_1 + 3x_2 + x_3$$

$$\text{sub to } -3x_1 + 2x_2 + 3x_3 = 8$$

$$-3x_1 + 4x_2 + 2x_3 = 7$$

$$x_1, x_2, x_3 \geq 0$$

(c) Use dominance to reduce the payoff matrix and solve the following game problem given by the payoff matrix :

		B	
	1	7	2
A	6	2	7
	0	1	6

P.T.O.

(d) Prove that, every extreme point of the convex set of all feasible solutions of the system

$$Ax = b, \; x \geq 0$$

corresponds to a basic feasible solution.

(e) Solve the following L.P.P. by two-phase simplex method :

Maximize $Z = 2x_1 + x_2 + x_3$

Subject to $4x_1 + 6x_2 + 3x_3 \leq 8$

$$3x_1 - 6x_2 - 4x_3 \leq 1$$

$$2x_1 + 3x_2 - 5x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0.$$

(f) Solve the following travelling salesman problem :

	A	B	C	D
A	∞	4	7	3
B	4	∞	6	3
C	7	6	∞	7
D	3	3	7	∞

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) Solve the following transportation problem :

	D_1	D_2	D_3	
O_1	8	7	3	60
O_2	3	8	9	70
O_3	11	3	5	80
	50	80	80	

(ii) Determine the position of the point (1, –2, 3, 4) relative to the hyperplane

$$4x_1 + 6x_2 + 2x_3 + x_4 = 2.$$ 8+2

(b) (i) Solve the following problem by two-phase method

$$\text{Max } Z = 5x_1 + 3x_2$$

$$\text{sub to } 2x_1 + x_2 \leq 1$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0.$$

(ii) Is assignment problem a Linear programming problem? Justify. 8+2

P.T.O.

(c) Give the dual of the following LPP and hence solve it :

Max $Z = 3x_1 - 2x_2$

sub to $x_1 \le 4$

$x_2 \le 6$

$x_1 + x_2 \le 5$

$-x_2 \le -1$

and $x_1, x_2 \ge 0.$ 10

(d) (i) If an L.P.P. has an optimal solution, then show that atleast one B.F.S. must be optimal.

(ii) Solve graphically the following L.P.P.

Maximize $Z = -x_1 + x_2$

Subject to $5x_1 + 6x_2 \ge 30$

$9x_1 - 2x_2 = 72$

$x_2 \le 9$

$x_1, x_2 \ge 0$

Also find its redundant constraint. 5+(4+1)

OR

[Point Set Topology]

Group - A

1. Answer any ***ten*** questions : 2×10=20

(a) Prove that $Fr(A)=\Phi$ if and only if A is both open and closed set.

(b) If every countable subset of X is closed, is the topology necessarily discrete?

(c) Show that the set of rational numbers is not locally compact.

(d) For what space X, the only dense set is X itself?

(e) Prove that the cardinal number of set of all continuous real valued functions on **R** is c.

(f) Prove that each path component of a space X is open if and only if each point of X has a path connected neighbourhood.

(g) Show that $[a, b]$ is homeomorphic to $[0, 1]$.

(h) Prove that ΠA_i is dense in ΠY_i if and only if $A_i \subset Y_i$.

(i) Show that intersection of two connected sets need not be connected. What about union? Justify your answer.

(j) Prove that the continuous image of a compact space is compact.

(k) Show that the subspace **Q** of rational numbers in the real line **R** is disconnected.

P.T.O.

(l) Define open map and give one example of it.

(m) Let $\tau_1 = \{\phi, \{1\}, X_1\}$ be a topology on $X_1 = \{1, 2, 3\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X_2\}$ be a topology on $X_2 = \{a, b, c, d\}$. Find a base for the product topology τ on $X_1 \times X_2$.

(n) Consider the topology $\tau = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ on $X = \{a, b, c, d\}$ and $Y = \{b, c, d\}$ is a subset of X. Then find relative topology on Y.

(o) State continuum hypothesis.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Show that a bijective map f from a topological space X onto a topological space Y is a homeomorphism if and only if f is a continuous and open map.

(b) Let X be a non-empty set and a mapping $\mathbf{I} : P(X) \to P(X)$ satisfies

(i) $\mathbf{I}(A) \subset A, \forall A \in P(X)$.

(ii) $\mathbf{I}(X) = X$.

(iii) $\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B), \forall\, A, B \in P(X)$.

(iv) $\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A), \forall\, A \in P(X)$.

Then show that $\tau = \{\mathbf{I}(A) : A \in P(X)\}$ is a topology and $\mathbf{I}(A) = int(A)$.

(c) Let

$\tau = \{\phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,e\}, \{a,b,c,d\}, X\}$

be a topology on $X = \{a, b, c, d, e\}$. Let $A = \{a, b, c\}$, $B = \{c, e\}$. Determine closure, exterior, interior, boundary, derived sets of each of the set A and B.

(d) Prove that the function $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

(e) Prove that the product topology in $\Pi_i Y_i$ is the smallest topology for which all projections $p_j : \Pi_i Y_i \to Y_j$ are continuous.

(f) Let (Y, τ_Y) be a subspace of a topological space (X, τ). Show that $\overline{A_Y} = Y \cap \overline{A}$.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) State and prove Lebesgue number theorem.

P.T.O.

(ii) State and prove Ascoli's theorem. 5+5

(b) (i) Let X be an uncountable set, and τ is the family consisting of empty set and all complements of countable sets. Show that τ is a topology on X.

(ii) Prove that $f : X \to Y$ is a open map if and only if $f(\text{int}\, A) \subset \text{int}(f(A))$ for each $A \subset X$, X and Y are topological spaces. 5+5

(c) (i) Prove that a topological space Y is locally connected if and only if the components of each open sets are open.

(ii) Let (X, τ) be a connected topological space and A be a connected subset of X and $X - A = G \cup H$, G and H be the separated sets. Prove that each of the sets $A \cup G$ and $B \cup H$ are connected. 5+5

(d) (i) Let (X, d) be a metric space and $A \subseteq X$. Then prove that if A is totally bounded, then A is bounded. Is the converse true? Justify your answer.

(ii) Prove that a compact subset of a metric space is closed and bounded. Is the converse true? Justify your answer. 5+5

OR

Group - A

[Theory of Equations]

1. Answer any ***ten*** questions from the following : 2×10=20

(a) Find a biquadratic equation with rational coefficients having $\left(\sqrt{2}\pm 1\right)$ as roots.

(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 - x^3 + 2x^2 + x + 1 = 0$, find the value of $\left(\alpha^3+1\right)\left(\beta^3+1\right)\left(\gamma^3+1\right)\left(\delta^3+1\right)$.

(c) Find the condition for which the equation $(x+1)^4 = a\left(x^4+1\right)$ is a reciprocal equation.

(d) If q, r, s be positive, then find the nature of the roots of the equation $x^4 + qx^2 + rx - s = 0$.

(e) Show that the equation $x^3 - 3x^2 - 9x + 27 = 0$ has a multiple root.

(f) If α be a multiple root of order 3 of the equation $x^4 + bx^2 + cx + d = 0 (d \neq 0)$, show that $\alpha = -\dfrac{8d}{3c}$.

P.T.O.

OR

Group - A

[Theory of Equations]

1. Answer any ***ten*** questions from the following : 2×10=20

(a) Find a biquadratic equation with rational coefficients having $(\sqrt{2} \pm 1)$ as roots.

(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 - x^3 + 2x^2 + x + 1 = 0$, find the value of $(\alpha^3+1)(\beta^3+1)(\gamma^3+1)(\delta^3+1)$.

(c) Find the condition for which the equation $(x+1)^4 = a(x^4+1)$ is a reciprocal equation.

(d) If q, r, s be positive, then find the nature of the roots of the equation $x^4 + qx^2 + rx - s = 0$.

(e) Show that the equation $x^3 - 3x^2 - 9x + 27 = 0$ has a multiple root.

(f) If α be a multiple root of order 3 of the equation $x^4 + bx^2 + cx + d = 0 (d \neq 0)$, show that $\alpha = -\dfrac{8d}{3c}$.

P.T.O.

(g) Show that the equation $1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}=0$ cannot have equal roots.

(h) Define Binomial equation and special root of an equation.

(i) If the roots of the equation $x^3+3px^2+3qx+r=0$ be in harmonic progression, then show that $2q^3=r(3pq-r)$.

(j) If x^4+px^2+qx+r has a factor of the form $(x-\alpha)^3$, prove that $8p^3+27q^2=0$.

(k) Let $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, where $a_0, a_1, \ldots, a_n$ are integers. If $\frac{p}{q}$ be a rational root of the equation $f(x)=0$, where p, q are prime to each other, then prove that p is a divisor of a_n.

(l) If the sum of two roots of the equation $x^3+\alpha x^2+\beta x+\gamma=0$ is zero, then find the relation among α, β, γ.

(m) Find an upper limit of the real roots of the equation $x^4-x^3-2x^2-4x+1=0$.

(n) Find the remainder when $x^5 - 3x^4 + 4x^2 + x + 4$ is divided by $(x+1)(x-2)$.

(o) The roots of the equation $x^3 - 3p.x^2 + 3(p-1)x + 1 = 0$ are α, β, γ, find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, $(r \neq 0)$ find an equation whose roots are

$$\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}, \frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\alpha}, \frac{1}{\alpha}+\frac{1}{\gamma}-\frac{1}{\beta}.$$

(b) Solve $x^3 - 6x^2 + 30x - 25 = 0$ by Cardan's method.

(c) If $f(x) = 2x^3 + 7x^2 - 2x - 3$, then express $f(x-1)$ as a polynomial in *x*. Apply Descartes's rule to both the equations $f(x) = 0$ and

P.T.O.

$f(-x)=0$ to determine the exact number of positive and negative roots of $f(x)=0$.

(d) Solve the equation $x^7+4x^6+4x^5+x^4-x^3-4x^2-4x-1=0$.

(e) Solve the equation $x^4-18x^2+32x-15=0$ by Ferrari's method.

(f) If α, β, γ be the roots of the equation $x^3+bx^2+cx+d=0$, $d \neq 0$, find the equation whose roots are $\alpha+\frac{1}{\alpha}$, $\beta+\frac{1}{\beta}$, $\gamma+\frac{1}{\gamma}$.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) Prove that the roots of the equation $\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+\cdots+\frac{A_n}{x+a_n}=x+b$ are all real, where a_i, A_i, b are all real numbers and $A_i > 0$.

(ii) If the equation $x^3+px^2+qx+r=0$ has a root $\alpha+i\alpha$, where p, q, r and α are real, prove that $\left(p^2-2q\right)\left(q^2-2pr\right)=r^2$. Hence, solve the equation $x^3-7x^2+20x-24=0$.

4+(3+3)

(b) (i) Show that the roots of the cubic $x^3 - 3x + 1 = 0$ are $2\sin 10°$, $2\sin 50°$ and $(-2\sin 70°)$.

(ii) If α be a non real root of the equation $x^7 - 1 = 0$, then find the equation whose roots are $(\alpha + \alpha^6)$, $(\alpha^2 + \alpha^5)$, $(\alpha^3 + \alpha^4)$. 5+5

(c) (i) Let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$, where $a_0, a_1, \ldots, a_n$ are integers. If $f(0)$ and $f(1)$ are both odd, prove that the equation can not have an integer root. Hence, prove that the equation $x^4 + 6x^3 + 3x^2 - 14x + 15 = 0$ can not have an integer root.

(ii) Reduce the equation $ax^3 + 3bx^2 + 3cx + d = 0$ (a, b, c, d are real and $a \neq 0$) to the standard form $z^3 + 3Hz + G = 0$ where G and H are to be determined by you. Hence obtain a necessary and sufficient condition in terms of G and H for the above equation have two equal roots.

(d) (i) Solve the equation $x^5 - 1 = 0$ and deduce the values of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$.

P.T.O.

(ii) Transform the equation $x^3 + 6x^2 + 9x + 4 = 0$ into one which shall want the second term and then find the equation whose roots are the squares of the differences of the roots of the obtained equation.